Math 429 - Exercise Sheet 1

1. For Lie groups consisting of matrices, one way to think about the connection between the Lie algebra $\mathfrak{g}$ and the Lie group G is via the procedure

$$X \in \mathfrak{g} \quad \rightsquigarrow \quad 1 + tX \in G \tag{1}$$

(at least for t small enough so that $1 + tX$ is invertible). Use this approach to show that the Lie algebras of $SL_n, O_n, Sp_{2n}, U(n), SU(n)$ are indeed the ones we listed in Subsection 1.6 of the notes.

2. It turns out that the notion (1) is simply the first order term of the correct one, which is given by the **exponential map**

$$X \in \mathfrak{g} \quad \rightsquigarrow \quad \exp(X) \in G \tag{2}$$

where

$$\left(\exp(X) = 1 + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \dots \right) \in GL_n \tag{3}$$

Check that for any X, (2) provides a **one-parameter subgroup**, i.e. a homomorphism

$$\mathbb{K} \to G, \qquad t \to \exp(tX)$$

where the LHS is made into a group with respect to addition, and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

3. For any matrices X, X', show that

$$\exp(tX)\exp(t'X')\exp(tX)^{-1}\exp(t'X')^{-1} = 1 + tt'[X, X'] + \text{higher order in } t, t'$$

so we can define the Lie bracket in terms of exponentials as

$$[X, X'] = \frac{\partial}{\partial t}\frac{\partial}{\partial t'} \exp(tX)\exp(t'X')\exp(tX)^{-1}\exp(t'X')^{-1}\Big|_{t=t'=0} \tag{4}$$

4. Let us now define the exponential map intrinsically, for any Lie group G with $\mathfrak{g} = T_eG$

$$\boxed{\exp : \mathfrak{g} \to G} \tag{5}$$

To do so, show that for any $x \in \mathfrak{g}$ there exists a unique one-parameter subgroup

$$\gamma_X : \mathbb{K} \to G \tag{6}$$

such that

$$\frac{\partial \gamma_X(t)}{\partial t}\Big|_{t=0} = X$$

Then we define

$$\boxed{\exp(X) = \gamma_X(1)} \tag{7}$$

Remark. *Consider the Lie group* $G = S^1$ *with the operation corresponding to complex multiplication. Its tangent space at the identity is* $\mathfrak{g} = \mathbb{R}$ *with the usual addition, and the exponential map is simply* $x \mapsto e^{2\pi i x}$*. Thus, we see that the exponential map may not be injective. It also may not be surjective in general, since by construction its image lies in the connected component of the identity in* G *(and disconnected Lie groups exist, e.g.* $O_n(\mathbb{R})$*; prove that this group is disconnected).*

5. For any Lie group G with Lie algebra $\mathfrak{g}$, show that the analogue of (4)

$$[X, X'] = \frac{\partial}{\partial t}\frac{\partial}{\partial t'} \exp(tX) \exp(t'X') \exp(tX)^{-1} \exp(t'X')^{-1}\Big|_{t=t'=0} \qquad (8)$$

(with the exponentials defined as in (7)) matches the Lie bracket defined in class on $\mathfrak{g}$ via left invariant vector fields.